PRINCIPLES OF ANALYSIS
LECTURE 18 - UNIFORM CONTINUITY

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1. Boundary Behavior

Recall that if $f : D \to \mathbb{R}$ is continuous and $x_0 \in D$ is an accumulation point of D, then f has a limit at x_0, and indeed $\lim_{x \to x_0} f(x) = f(x_0)$. However, if x_0 is not in D, other possibilities exist.

Let $D = (-1, 1)$ and $x_0 = 1$; clearly, x_0 is an accumulation point of D. Let $f : D \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1-x^2}$. Now f is continuous on D and x_0 is an accumulation point of D, but the limit does not exist at x_0.

This happens because continuity is a *local property*, as opposed to a *global property*. A local property is one which does not look beyond some neighborhood of every point, no matter how small that neighborhood may be.

For example, whether or not a set is open is a local property of the set, but whether or not it is bounded is a global property.

2. Uniform Continuity

Let $E \subset \mathbb{R}$ and let $f : E \to \mathbb{R}$. We say that f is *uniformly continuous* on E if

$$\forall \epsilon > 0 \, \exists \delta > 0 \ni |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon,$$

where $x, y \in E$.

Proposition 1. *Let $f : D \to \mathbb{R}$ be uniformly continuous and let x_0 be an accumulation point of D. Then f has a limit at x_0.*

Proof. Recall that f has a limit at x_0 if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in D which converges to x_0, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in D which converges to x_0; to show that $\{f(x_n)\}_{n=1}^{\infty}$ converges, it suffices to show that it is a Cauchy sequence.

Let $\epsilon > 0$; we wish to find $N \in \mathbb{Z}^+$ such that if $m, n \geq N$, then $|f(x_m) - f(x_n)| < \epsilon$. Since f is uniformly continuous, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$, where $x, y \in E$.

Since $\lim_{n \to \infty} x_n = x_0$, there exists $N \in \mathbb{Z}^+$ such that $|x_n - x_0| < \frac{\delta}{2}$ whenever $n \geq N$. Then for $m, n \geq N$, we have $|x_m - x_n| < \delta$, so $|f(x_m) - f(x_n)| < \epsilon$. Since ϵ was selected arbitrarily, this shows that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, and is therefore convergent. $\square$

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Proposition 2. *Let $f : E \to \mathbb{R}$ be a continuous function. If E is compact, then f is uniformly continuous.*

Proof. Suppose that E is compact.

Let $\epsilon > 0$; we wish to find $\delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Since f is continuous, then for every $x \in E$ there exists $\delta_x > 0$ such that if $y \in E$ and $|x - y| < \delta_x$, then $|f(x) - f(y)| < \epsilon$.

Let $V_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$; this is an open set which contains x. Let $\mathcal{C} = \{V_x \mid x \in E\}$; then $\mathcal{C}$ is an open cover of E. Since E is compact, there exists a finite subcover, so there exist $x_1, \dots, x_n \in E$ such that $E \subset \cup_{i=1}^n V_{x_n}$. Set

$$\delta = \min\{\delta_{x_i}/2 \mid i = 1, \dots, n\}/2.$$

Let $x, y \in E$ with $|x - y| < \delta$. Now there exists x_i such that $|x - x_i| < \frac{\delta_{x_i}}{2}$. Then

$$\begin{aligned} |y - x_i| &= |y - x + x - x_i| \\ &\leq |y - x| + |x - x_i| \\ &< \delta + \frac{\delta_{x_i}}{2} \\ &\leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} \\ &= \delta. \end{aligned}$$

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